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A rational spectral collocation method for solving a class of parameterized singular perturbation problems

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ABSTRACT

A new kind of numerical method based on rational spectral collocation with the sinh transformation is presented for solving parameterized singularly perturbed two-point boundary value problems with one boundary layer. By means of the sinh transformation, the original Chebyshev points are mapped onto the transformed ones clustered near the singular points of the problem. The results from asymptotic analysis as regards the singularity of the solution are employed to determine the parameters in the transformation. Numerical experiments including several nonlinear cases illustrate the high accuracy and efficiency of our method.

1. Introduction

In this paper, a class of parameterized singular perturbed boundary value problems (PSPBVP) are considered, as follows:

$$\varepsilon \frac{dy}{dx} + f(x, y, \lambda) = 0, \quad x \in [0, 1] \tag{1.1}$$

$$y(0) = A, \qquad y(1) = B \tag{1.2}$$

where ε is a given small positive parameter, and A and B are given constants. The function $f(x, y, \lambda)$ is assumed to be such that

$$f(x, y, \lambda) \in C^3([0, 1] \times \mathbb{R}^2) \tag{1.3}$$

$$\frac{\partial f}{\partial y}(x, y, \lambda) \geq a_0 > 0, \quad (x, y, \lambda) \in ([0, 1] \times \mathbb{R}^2) \tag{1.4}$$

$$\frac{\partial f}{\partial \lambda}(x, y, \lambda) \neq 0, \quad (x, y, \lambda) \in ([0, 1] \times \mathbb{R}^2). \tag{1.5}$$

By a solution of (1.1)–(1.2) we mean a pair $\{y(x), \lambda\} \in C^1([0, 1]) \times \mathbb{R}$ for which (1.1)–(1.2) is satisfied. Under the assumptions of (1.3)–(1.5), the solution $y(x)$, in general, has a boundary layer of width $\mathcal{O}(\varepsilon)$ near $x = 0$ for $\varepsilon \ll 1$.

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Parameterized problems have been considered for many years. For discussion of the existence, uniqueness and applications of parameterized equations see [1–6] and references therein. However, the algorithms and approximating results in the above mentioned papers are only concerned with cases without boundary layers.

The numerical analysis of parameterized singular perturbation cases has always suffered from more difficulties due to the boundary layer behavior of the solution as well as the nonlinearity of the problem. Recent years have witnessed substantial progress in the development of numerical methods for solving such problems. Amiraliyev et al. gave a finite difference method on the Shishkin mesh [7] and the Bakhvalov mesh [8]. Feng Xie et al. [9] put forward a so-called novel method in which a boundary layer correction technique and a Runge–Kutta scheme are employed.

Here we present a new kind of numerical method based on a rational spectral collocation method with the sinh transformation (the RSC–sinh method), combining with some results from asymptotic analysis of PSPBVP and the idea of linearization.

It is well known that spectral methods are attractive in solving singularly perturbed problems thanks to the fact that the spectral collocation points, for example, the original Chebyshev points $x_k = \cos(k\pi/N)$, are clustered at the boundary; more precisely, we have

$$|x_1 - x_0| = |x_N - x_{N-1}| = |\cos(\pi/N) - 1| \approx \frac{1}{2}\left(\frac{\pi}{N}\right)^2 \approx \frac{5}{N^2}.$$

In other words, the spacing between the collocation points near the boundaries is $\mathcal{O}(N^{-2})$, in contrast with $\mathcal{O}(N^{-1})$ for finite differences or finite elements methods.

Although spectral methods are much more efficient than using finite differences and finite elements in solving for boundary layers, still a large N is required to obtain accurate solutions when ε is small. In the past few years, in order to resolve very thin boundary layers problems using a reasonable value of N, several modified spectral methods have been proposed [10–12]. A typical idea is that on introducing some transformation $x \mapsto x^t = t(x)$, the transformed collocation points x^t become more clustered in the boundary layer region, which is also called "coordinate stretching". A problem of this idea in practice is that after such a transformation, the derivatives in the underlying differential equation (i.e. the $\frac{dy}{dx}$ in (1.1)) have to be transformed into new coordinates (i.e. $\frac{dy}{dx^t} = \frac{1}{t'(x)}\frac{dy}{dx}$), which would greatly increase the computational complexity.

A significant advantage of the spectral collocation method based on rational interpolants in barycentric form, proposed by Berrut and collaborators [13,14], is that after the transformation, the derivatives in the underlying differential equation do not have to be transformed correspondingly, as is usual in other methods. The improved RSC–sinh method has been developed by Tee and Trefethen [12]. They devised a sinh transformation that maps original Chebyshev points clustered near the boundaries of $[-1, 1]$ into a new set of collocation points, say the transformed Chebyshev points, which are clustered near the singular point of a function. Since the traits of this method are well suited for addressing singular perturbed problems, we employ it to solve the PSPBVP numerically.

In order to determine the parameters in the above sinh transformation, the singularities of the solution, including the location and width of the boundary layer, have to be known. However, it is rather a difficult problem that the singularities of the solution are usually not known exactly. Hence, it is necessary to approximate them using some singularity location technique. Fortunately, for a class of PSPBVP like (1.1)–(1.2), the asymptotic results recently obtained in [9] can give us enough information about the singularities of the solution. Besides, if $f(x, y, \lambda)$ in (1.1) is a nonlinear function of y, the linearization technique based on Taylor's expansion will be used.

Numerical experiments illustrate that the RSC–sinh method enjoys improved spectral accuracy while the increase in computing cost is slight.

This paper is organized as follows. The asymptotic analysis of the problem is presented in Section 2. In Section 3, we introduce the RSC–sinh method. In Section 4, we explain how the new method can be implemented from the ideas discussed in Sections 2 and 3, and give the numerical algorithms in detail. The numerical results for several nonlinear examples are given in Section 5. Finally in Section 6, we present some conclusions.

2. Analytic results

The following two theorems and their proofs are given in [9]. Here we only present the conclusions available in order to take full advantage of them in following sections.

Theorem 2.1. *Under the assumptions of (1.3)–(1.5), the PSPBVP (1.1)–(1.2) has a unique solution $\{y(x), \lambda\} \in C^1([0, 1]) \times \mathbb{R}$ satisfying*

$$y(x) = u(x) + v(\tau) + \mathcal{O}(\varepsilon), \qquad \lambda = \lambda_0 + \mathcal{O}(\varepsilon), \quad \tau = \frac{x}{\varepsilon}$$

where $u(x)$ is the solution of the reduced problem obtained by setting $\varepsilon = 0$ in (1.1) and with the boundary condition on $x = 1$

$$\begin{cases} f(x, y, \lambda) = 0 \\ y(1) = B \end{cases} \tag{2.1}$$

and $v(\tau)$ is the solution of boundary layer corrected equation

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}\tau} + f(0, u(0) + v(\tau), \lambda_0) = 0 \\ v(0) = A - u(0) \end{cases} \tag{2.2}$$

where λ_0 can be also obtained from (2.1) at the same time as $u(0)$.

Remark 2.1. According to the singular perturbation theory, over most of the interval $[0, 1]$, the solution of the reduced problem $y = u(x)$ approximates the solution of (1.1)–(1.2). Hence, we will employ $y = u(x)$ as the initial value in the process of linearization iteration in step 4 in Section 4.

Theorem 2.2. *The solution of (1.1)–(1.2) has the following asymptotic expansion:*

$$y(x) = u(x) + \bar{v}(\tau) + \mathcal{O}(\varepsilon), \quad \tau = \frac{x}{\varepsilon}$$

where

$$\bar{v}(\tau) = \begin{cases} v(\tau), & 0 \le \tau \le T \\ 0, & \tau > T, \end{cases} \qquad T = \frac{1}{a_0} \ln \frac{|A - u(0)|}{\varepsilon} \tag{2.3}$$

in which $v(\tau)$ is the solution of (2.2).

Remark 2.2. The above theorem suggests that the boundary layer region of the solution $y(x, \varepsilon)$ is $[0, \varepsilon T]$, that is to say, the location of the boundary layer is at the left endpoint of interval $[0, 1]$ and its width is εT. This result plays an important role in the RSC–sinh method in Section 3.

3. The RSC–sinh method

3.1. Transformed Chebyshev points

The original Chebyshev points are

$$x_k = \cos(k\pi/N) \in [-1, 1], \quad k = 0, 1, \ldots, N.$$

These points are clustered near the boundaries of $[-1, 1]$. However, what we seek are the points that are denser inside the boundary layer region than outside. To achieve this goal, we consider the transformation given in [12]

$$g_{\alpha,\beta}(x) = \alpha + \beta \sinh\left[\left(\sinh^{-1}\left(\frac{1-\alpha}{\beta}\right) + \sinh^{-1}\left(\frac{1+\alpha}{\beta}\right)\right)\frac{x-1}{2} + \sinh^{-1}\left(\frac{1-\alpha}{\beta}\right)\right] \tag{3.1}$$

where α, β are parameters dependent on the singularity of the solution, which respectively represent the location and width of the boundary layer.

The transformed Chebyshev points are

$$x_k^g = g_{\alpha,\beta}\left(\cos\left(k\pi/N\right)\right) \in [-1, 1], \quad k = 0, 1, \ldots, N.$$

These points are clustered near the location of the boundary layer $x = \alpha$ and their density is determined by the boundary layer width β. The thinner the boundary layer, the denser the points.

For the PSPBVP (1.1)–(1.2), we chose $\alpha = -1$, $\beta = 2\varepsilon T$; then

$$x_k^g = g_{-1,2\varepsilon T}\left(\cos\left(k\pi/N\right)\right) \in [-1, 1]. \tag{3.2}$$

With a map from $[-1, 1]$ to $[0, 1]$, we can get

$$\hat{x}_k = 0.5\left(x_k^g + 1\right) \in [0, 1]. \tag{3.3}$$

The points $\{\hat{x}_k\}_{k=0}^N$ are clustered near $x = 0$ with width εT, which matches well with the boundary layer behavior discussed in Remark 2.2.

After the mapping in (3.3), the underlying differential equation (1.1) will be correspondingly transformed to

$$2\varepsilon \frac{\mathrm{d}y}{\mathrm{d}x^g} + \tilde{f}(x^g, y, \lambda) = 0, \quad x^g \in [-1, 1] \tag{3.4}$$

where $\tilde{f}(x^g, y, \lambda) = f(0.5(x^g + 1), y, \lambda)$.

As to the mapping in (3.2), a similar transformation of the derivatives in the underlying differential equation (1.1) will not be required when the RSC–sinh method based on rational interpolants in barycentric form is employed.

3.2. The differentiation matrix

The barycentric form of a rational function $r_N(x)$ which interpolates the function $h(x)$ between the points $\{\hat{x}_k\}$ is [13]

$$r_N(x) = \frac{\sum_{k=0}^{N} \frac{\omega_k}{x-\hat{x}_k} h(\hat{x}_k)}{\sum_{k=0}^{N} \frac{\omega_k}{x-\hat{x}_k}} \tag{3.5}$$

where the barycentric weights $\{\omega_k\}$ are $\omega_0 = \frac{1}{2}$, $\omega_N = \frac{(-1)^N}{2}$, $\omega_k = (-1)^k$, $k = 0, 1, \dots, N-1$.

The rational interpolation based on the barycentric form with transformed Chebyshev points has the following convergence analysis.

Theorem 3.1 ([14]). *Let D_1, D_2 be domains in $\mathbb{C}$ containing $J = [-1, 1]$ and a real interval I respectively. Let $g : D_1 \mapsto D_2$ be a conformal map such that $g(J) = I$. Suppose that $h : D_2 \mapsto \mathbb{C}$ is a function such that the composition $h \circ g : D_1 \mapsto \mathbb{C}$ is analytic inside and on an ellipse E_ρ, $\rho > 1$, with foci at ± 1 and the sum of its semi-major axis length and semi-minor axis length equal to ρ. Let $r_N(x)$ be the rational function (3.5) interpolating $h(x)$ between transformed Chebyshev points $\hat{x}_k = g(\cos(k\pi/N))$ with barycentric weights. For $\forall x \in [-1, 1]$,*

$$|r_N(x) - h(x)| = \mathcal{O}(\rho^{-N}).$$

Remark 3.1. Theorem 3.1 suggests that one should choose a conformal map g (i.e. (3.1)) so that the ellipse of analyticity of $h \circ g$ is larger than h, and apply g in a spectral method based on a rational interpolant of the form (3.5), to obtain an approximation of h which is more accurate than that obtained using the Chebyshev spectral method with the same number of grid points.

An advantage of representing a rational interpolation in barycentric form is that its derivatives can be evaluated easily using differentiation formulae derived in [15], instead of using the quotient rule repeatedly. The nth derivative of $r_N(x)$ evaluated at $\hat{x}_j$ can be written in the form $r_N^{(n)}(\hat{x}_j) = \sum_{k=0}^{N} D_{jk}^{(n)} h(\hat{x}_k)$, where the entries of $D^{(1)}$ and $D^{(2)}$, the first- and second-order differentiation matrices, are given by

$$D_{jk}^{(1)} = \begin{cases} \dfrac{\omega_k}{\omega_j(\hat{x}_j - \hat{x}_k)}, & j \neq k \\ -\sum\limits_{i \neq k} D_{ji}^{(1)}, & j = k \end{cases} \tag{3.6}$$

$$D_{jk}^{(2)} = \begin{cases} 2D_{jk}^{(1)} \left(D_{jj}^{(1)} - \dfrac{1}{\hat{x}_j - \hat{x}_k} \right), & j \neq k \\ -\sum\limits_{i \neq k} D_{ji}^{(2)}, & j = k. \end{cases} \tag{3.7}$$

The derivatives of function $h(x)$ at the points $\left\{\hat{x}_k\right\}_{k=0}^{N}$ can be evaluated as the product of differentiation matrices and data vectors, as the common spectral collocation method does.

Remark 3.2. Note that the differentiation matrices in (3.6) and (3.7) rely just on the barycentric weights ω and new grid points $\hat{x}_k$. Hence, the transformation of the derivatives in the underlying equation into new coordinates is not required after the mapping in (3.2).

4. The numerical algorithm

In this section, we will present our method for solving PSPBVP (1.1)–(1.2) in detailed algorithms including four practical steps, which are the applications of the idea shown in Sections 2 and 3.

Step 1: In order to approximate the $u(0)$ in (2.3), the reduced problem (2.1) should be solved by means of the following Newton–Raphson iteration:

$$\lambda^{(k+1)} = \lambda^{(k)} - \frac{f\left(1, B, \lambda^{(k)}\right)}{(\partial f/\partial \lambda)\left(1, B, \lambda^{(k)}\right)} \tag{4.1}$$

$$u_i^{(k+1)} = u_i^{(k)} - \frac{f\left(x_i, u_i^{(k)}, \lambda^{(k+1)}\right)}{(\partial f/\partial u)\left(x_i, u_i^{(k)}, \lambda^{(k+1)}\right)} \tag{4.2}$$

where u_i denotes the approximation of $u(x)$ at $x = x_i$; $\{x_i\}_{i=0}^{I}$ can be selected as equally spaced points on the interval $[0, 1]$, $x_0 = 0$, $x_I = 1$; $u_i^{(0)}$ and $\lambda^{(0)}$ are the initial iteration values. (We set $\lambda^{(0)} = 0$, $u_i^{(0)} = 0$ in the examples of the next section.) The results of the iteration are denoted as λ_0 and u_i.

Table 1
Comparison of the errors of Example 1.

ε	RSC–sinh method		Method of [9]	
	Absolute error	Relative error	Absolute error	Relative error
1e−3	8.88e−16	9.01e−16	7.66e−02	1.85e−01
1e−4	4.55e−15	4.67e−15	7.66e−02	1.88e−01
1e−5	3.76e−13	3.92e−13	7.65e−02	1.79e−01
1e−6	7.28e−12	7.73e−12	7.66e−02	1.90e−01
1e−7	8.64e−11	9.39e−11	7.65e−02	1.95e−01
1e−8	5.12e−10	5.71e−10	7.66e−02	1.92e−01
1e−9	2.51e−08	2.89e−08	7.66e−02	1.82e−01
1e−10	3.89e−07	4.63e−07	7.63e−02	2.00e−01
1e−11	2.70e−06	3.33e−06	7.65e−02	1.78e−01
1e−12	1.74e−05	2.25e−05	7.65e−02	1.93e−01
1e−13	2.58e−05	3.49e−05	7.59e−02	2.07e−01
1e−14	5.12e−04	7.26e−04	7.62e−02	1.72e−01
1e−15	2.70e−03	3.90e−03	7.65e−02	1.83e−01
1e−16	2.86e−02	4.99e−02	7.64e−02	1.93e−01

Step 2: Compute the following variables in an orderly fashion: T using (2.3) in which $u(0) = u_0$, x_k^g using (3.2), $\hat{x}_k$ using (3.3) and $D^{(1)}$ using (3.6).

Step 3: Compute $\hat{u}_k$ in the same way as in (4.2), in which x_i should be replaced by $\hat{x}_k$.

Step 4: All three steps above have focused on the reduced problem, in which λ and $\hat{u}_k$, $D^{(1)}$ corresponding to $\hat{x}_k$ have been obtained. In this step, we will resolve the original problem. In the first substep, the RSC–sinh method is applied to (1.1):

$$2\varepsilon D^{(1)}\hat{y} + \tilde{f}(\hat{x}, \hat{y}, \lambda_0) = 0 \tag{4.3}$$

where $\hat{x}$, $\hat{y}$ are both vectors. Combining with the boundary condition $y(0) = A$, (4.3) can be solved.

In the second substep, if $f(x, y, \lambda)$ in (1.1) is a nonlinear function of y, then it can be linearized via Taylor's expansion as follows:

$$f\left(x, y^{(k+1)}, \lambda\right) \approx f\left(x, y^{(k)}, \lambda\right) + \frac{\partial f\left(x, y^{(k)}, \lambda\right)}{\partial y}\left(y^{(k+1)} - y^{(k)}\right). \tag{4.4}$$

Summing the two substeps (4.3) and (4.4), we can get the iteration process:

$$2\varepsilon D^{(1)}y^{(k+1)} + \tilde{f}\left(\hat{x}, y^{(k)}, \lambda_0\right) + \frac{\partial \tilde{f}\left(\hat{x}, y^{(k)}, \lambda_0\right)}{\partial y}\left(y^{(k+1)} - y^{(k)}\right) = 0 \tag{4.5}$$

in which the initial value $y^{(0)}$ should be selected as $\hat{u}(x)$, the solution of the reduced problem (2.1), which is approximated by $\hat{u}(x_k) \approx \hat{u}_k$ in Step 3. If the norm of $\left(y^{(k+1)} - y^{(k)}\right)$ is less than the tolerated error, the iteration could be stopped. In the following examples, by choosing the tolerated error to be approaching the machine epsilon (i.e. 1e−15), the error of iteration is essentially guaranteed to be harmless.

5. Examples

To demonstrate the high accuracy and efficiency of our method, numerical experiments including nonlinear PSPBVP with a left-end boundary layer are performed.

Example 1. Consider the following nonlinear problem:

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}y}{\mathrm{d}x} + y - \mathrm{e}^{-y} + (x+\lambda)\,\mathrm{e}^{-1/\varepsilon} + \mathrm{e}^{\left(x\mathrm{e}^{-1/\varepsilon} - \mathrm{e}^{-x/\varepsilon}\right)} + \mathrm{e}^{\lambda} + \lambda + 1 = 0 \\ y(0) = 1, \qquad y(1) = 0. \end{cases}$$

When λ satisfies $(\lambda - \varepsilon)\mathrm{e}^{-1/\varepsilon} + \mathrm{e}^{\lambda} + \lambda - 1 = 0$, this example has the exact solution

$$y(x) = \mathrm{e}^{-x/\varepsilon} - x\mathrm{e}^{-1/\varepsilon}.$$

Feng Xie [9] solved this problem using the boundary layer correction technique and a Runge–Kutta scheme. Table 1 illustrates the absolute errors and relative errors obtained by our method and the method in [9], in which $N = 70$ and ε varies from 1e −3 to 1e−16.

Tao Tang et al. proposed two kinds of spectral methods combined with some suitable transformation. One family is that of the proposed Legendre–Galerkin method (PLGM) with the following transformation [11]:

$$x^t = g_k(x) = -1 + \kappa \int_{-1}^{x} (1-\eta^2)^k \mathrm{d}\eta, \quad k \geq 1, \text{ and } \kappa = \frac{2}{\int_{-1}^{1}(1-\eta^2)^k \mathrm{d}\eta}. \tag{5.1}$$

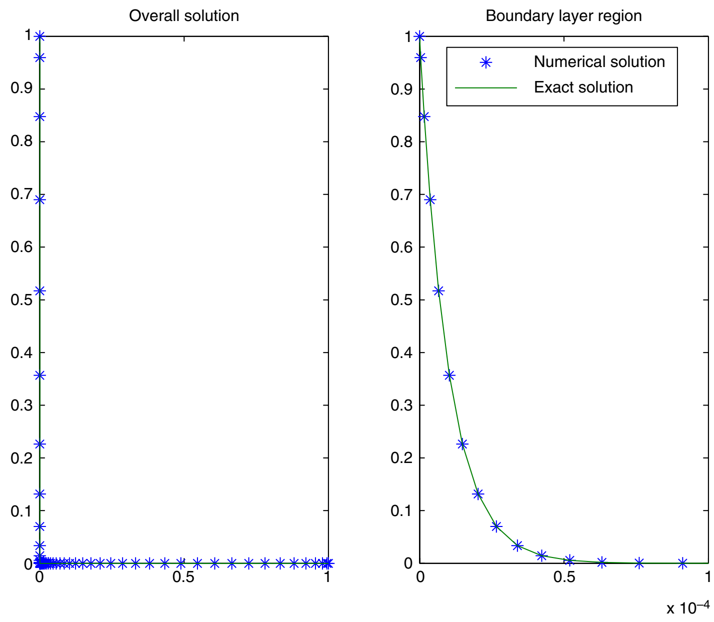

Fig. 1. Numerical solution for Example 1 obtained by the RSC–sinh method with $N = 70$.

Table 2
Comparison of the CPU times ($N = 70$ in Example 1).

ε	RSC–sinh	PLGM	BLRCC	Method of [9]
1e−3	0.0083	0.0100	0.0078	0.0072
1e−5	0.0085	0.0109	0.0075	0.0074
1e−8	0.0085	0.0110	0.0081	0.0079

Another family is that of the boundary layer resolving Chebyshev collocation method (BLRCC) with the transformation $x^t = g_m(x)$ where [10]

$$g_0(x) = x, \qquad g_m(x) = \sin\left(\frac{\pi}{2} g_{m-1}(x)\right), \quad m \geq 1. \tag{5.2}$$

In Eqs. (5.1) and (5.2), x is the original variable (i.e. the original Chebyshev point) and x^t is the transformed variable (i.e. the transformed Chebyshev point).

We employ the conventional Chebyshev collocation method (CCC), PLGM, BLRCC and the RSC–sinh method to solve this problem (in fact to solve (4.5)).

Figs. 1 and 2 show the numerical solutions obtained by the RSC–sinh and BLRCC approaches respectively. There are more points located in the boundary layer region in Fig. 1 than in Fig. 2, though the total number of collocation points is $N = 70$ in both of the methods. The reason is that after transformation (5.2), the points are clustered towards both of the endpoints of the underlying interval, while after transformation (3.1) the points are clustered near the location of the boundary layer (i.e. the left endpoint of the interval).

In Figs. 3–5, we plot the maximum errors on a semi-log scale, in the cases with $\varepsilon = 1\text{e}{-3}, 1\text{e}{-5}, 1\text{e}{-8}$ respectively. These figures show that all of the transformations (3.1), (5.1) and (5.2) could speed up the spectral convergence, but the RSC–sinh method is able to achieve higher accuracy while using fewer points.

Table 2 illustrates the comparison of CPU times, which suggests that all of the methods presented here have almost the same computing cost when applied to this problem.

Example 2. Consider the problem [8]

$$\begin{cases} \varepsilon \dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y - \mathrm{e}^{-y} + x^2 + \lambda + \tanh(x+\lambda) = 0 \\ y(0) = 1, \qquad y(1) = 0. \end{cases}$$

The exact solution of Example 2 is not available. Figs. 6 and 7 show the plots of the numerical solutions obtained by both our method and the method of [9] using $N = 100$ for the cases with $\varepsilon = 1\text{e} - 5, 1\text{e} - 15$ respectively; they indicate the validity of our method.

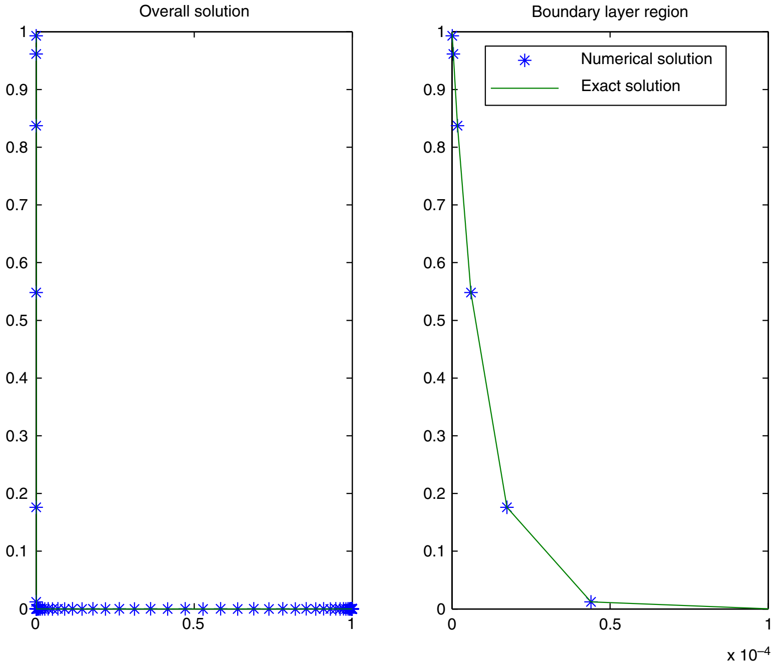

Fig. 2. Numerical solution for Example 1 obtained by the BLRCC method with $N = 70$.

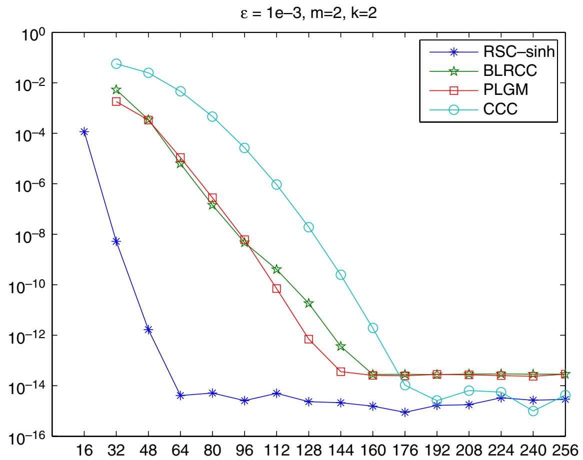

Fig. 3. Exponential rate of convergence when $\varepsilon = 1\text{e}{-3}$.

6. Conclusion

A new kind of numerical method for solving PSPBVP, called the RSC–sinh method, has been proposed. The key to the success of this method is applying the results from asymptotic analysis in deciding the parameters depending on the singularity of the solution to the sinh transformation.

Numerical experiments show that compared to the classic spectral method with other transformations, the present RSC–sinh method employed for solving PSPBVP has the following advantages:

1. The RSC–sinh method enjoys higher accuracy but using fewer collocation points.
2. The transformed collocation points in the RSC–sinh method are clustered near the location of the boundary layer, which is suitable for addressing this PSPBVP with only one boundary layer, while the transformation as in (5.2) could make the points clustered near both of the endpoints of the underlying interval, which may be more suitable for other problems with two boundary layers.
3. Since the RSC method is based on rational interpolants in barycentric form, after transformation, the derivatives in the underlying differential equation do not have to be transformed correspondingly as is usual in other methods.

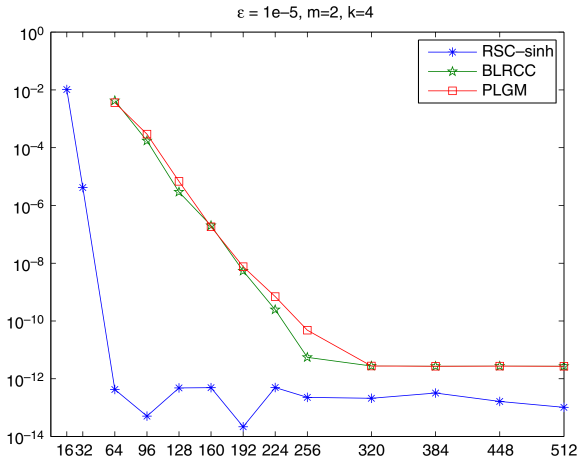

Fig. 4. Exponential rate of convergence when $\varepsilon = 1\text{e}{-5}$.

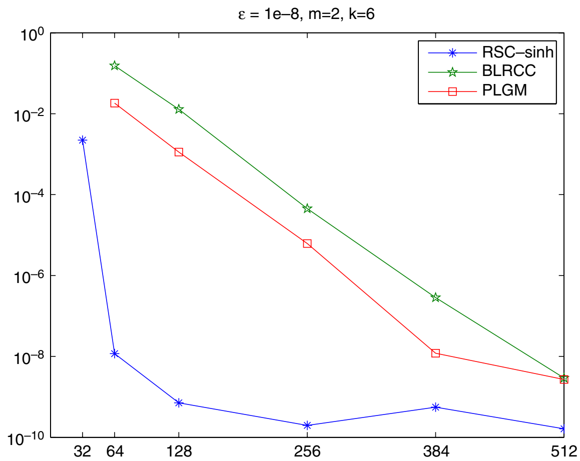

Fig. 5. Exponential rate of convergence when $\varepsilon = 1\text{e}{-8}$.

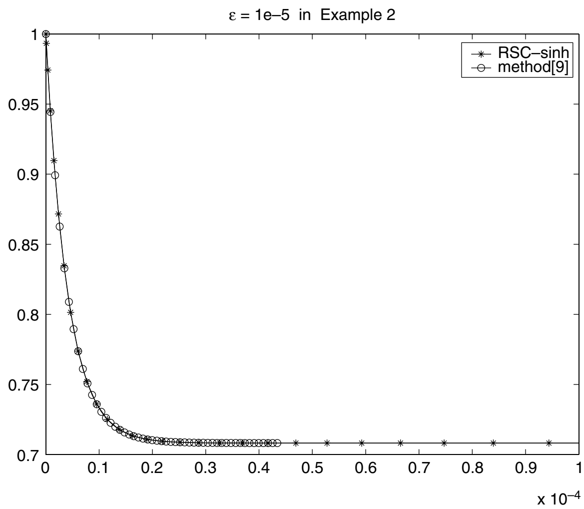

Fig. 6. Numerical solution of Example 2 with $\varepsilon = 1\text{e} - 5$.

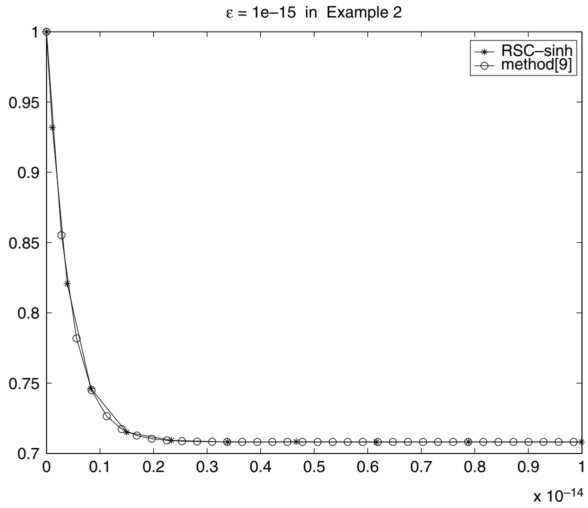

Fig. 7. Numerical solution of Example 2 with $\varepsilon = 1\text{e} - 15$.

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